

# From semigroups to subelliptic estimates for quadratic operators

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**Abstract:** Using an approach based on the techniques of FBI transforms, we give a new simple proof of the global subelliptic estimates for non-selfadjoint non-elliptic quadratic differential operators, under a natural averaging condition on the Weyl symbols of the operators, established by the second author [18]. The loss of the derivatives in the subelliptic estimates depends directly on algebraic properties of the Hamilton maps of the quadratic symbols. Using the FBI point of view, we also give accurate smoothing estimates of Gelfand-Shilov type for the associated heat semigroup in the limit of small times.

**Keywords and Phrases:** Non-selfadjoint operator, subelliptic estimate, quadratic differential operator, heat semigroup, FBI transform.

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## 1 Introduction and statement of results

There has recently been a large number of new developments for non-selfadjoint quadratic differential operators. Here some of the motivation comes from the analysis of second-order operators of Kramers-Fokker-Planck type, where non-selfadjoint non-elliptic quadratic operators often arise as local models via harmonic oscillator approximation, [6], [7], [9], [15]. The recent results in the quadratic case include precise bounds on the resolvent and spectral projections, [12], [24], determination of spectra for partially elliptic operators [8], as well as smoothing and decay estimates for the corresponding semigroup in the limit of large times, [8], [16], [1], [4]. Of particular relevance for the present paper is the work [18] by the second author, where global subelliptic estimates are established for the class of non-selfadjoint non-elliptic quadratic differential operators, whose Weyl symbols enjoy certain dynamical averaging properties, studied in [8]. The purpose of this work is to develop a new time-dependent approach to the results of [18]. When doing so, we shall also establish a precise form of the Gelfand-Shilov regularizing property for the associated quadratic semigroup, in the small time limit. Let us proceed now to describe the assumptions, state the results, and outline the main ideas of the proofs.

Let  $q$  be a complex valued quadratic form on the phase space,

$$q : \mathbf{R}_x^n \times \mathbf{R}_\xi^n \rightarrow \mathbf{C}, \quad (x, \xi) \mapsto q(x, \xi). \quad (1.1)$$

We shall assume throughout the following discussion that the quadratic form  $\operatorname{Re} q$  is positive semi-definite,

$$\operatorname{Re} q(x, \xi) \geq 0, \quad (x, \xi) \in \mathbf{R}^{2n}. \quad (1.2)$$

Associated to  $q$  is the Hamilton map,

$$F : \mathbf{C}^{2n} \rightarrow \mathbf{C}^{2n}, \quad (1.3)$$

defined by the identity

$$q(X, Y) = \sigma(X, FY), \quad X, Y \in \mathbf{C}^{2n}. \quad (1.4)$$

Here  $\sigma$  is the complex symplectic form on  $\mathbf{C}^{2n}$  and the left hand side is the polarization of  $q$ , viewed as a symmetric bilinear form on  $\mathbf{C}^{2n}$ . We notice that the Hamilton map  $F$  is skew-symmetric with respect to  $\sigma$ .

Following [8], [18], let us introduce the so called singular space associated to  $q$ ,

$$S = \left( \bigcap_{j=0}^{2n-1} \text{Ker} [\text{Re } F(\text{Im } F)^j] \right) \bigcap \mathbf{R}^{2n}. \quad (1.5)$$

As established in [8, Section 2], an equivalent description of the linear subspace  $S \subset \mathbf{R}^{2n}$  can be given as follows,

$$S = \left\{ X \in \mathbf{R}^{2n}; \ H_{\text{Im } q}^k \text{Re } q(X) = 0, \ k \in \mathbf{N} \right\}. \quad (1.6)$$

Here  $H_f = f'_\xi \cdot \partial_x - f'_x \cdot \partial_\xi$  is the Hamilton vector field of a function  $f \in C^1(\mathbf{R}_{x,\xi}^{2n}; \mathbf{R})$ . Throughout this work, it will be assumed that the singular space of  $q$  is trivial,

$$S = \{0\}. \quad (1.7)$$

It was shown in [8, Proposition 2.0.1] that the assumption (1.7) implies that for each  $T > 0$ , the quadratic form

$$\mathbf{R}^{2n} \ni X \mapsto \int_0^T \text{Re } q(\exp(tH_{\text{Im } q})(X)) \, dt$$

is positive definite.

In view of (1.7), we may introduce  $0 \leq k_0 \leq 2n - 1$  to be the smallest integer such that

$$\left( \bigcap_{j=0}^{k_0} \text{Ker} [\text{Re } F(\text{Im } F)^j] \right) \bigcap \mathbf{R}^{2n} = \{0\}. \quad (1.8)$$

Let us recall from [8, Section 2], [9] that (1.8) implies the following subelliptic condition for the quadratic symbol  $q$ : for each  $0 \neq X \in \mathbf{R}^{2n}$ , there exists an integer  $j \in \{0, \dots, k_0\}$  such that

$$\operatorname{Re} q(\exp(tH_{\operatorname{Im} q})(X)) = at^{2j} + \mathcal{O}(t^{2j+1}), \quad t \rightarrow 0, \quad (1.9)$$

where  $a = (H_{\operatorname{Im} q}^{2j} \operatorname{Re} q)(X)/(2j)! > 0$ .

The following result was established in [18], where we write

$$q^w(x, D_x)u(x) = \frac{1}{(2\pi)^n} \iint e^{i(x-y) \cdot \xi} q\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi, \quad u \in \mathcal{S}(\mathbf{R}^n), \quad (1.10)$$

for the Weyl quantization of  $q$ . Before stating it, let us introduce the unbounded selfadjoint operator

$$\langle (x, D_x) \rangle^r = (1 + x^2 + D_x^2)^{r/2}, \quad r > 0,$$

defined in terms of the functional calculus for the selfadjoint operator  $D_x^2 + x^2$  on  $L^2(\mathbf{R}^n)$ . Here  $D_x = i^{-1}\partial_x$ .

**Theorem 1.1** *Let  $q : \mathbf{R}_x^n \times \mathbf{R}_\xi^n \rightarrow \mathbf{C}$  be a quadratic form with  $\operatorname{Re} q \geq 0$ , such that (1.7) holds. Let  $k_0 \in \{0, \dots, 2n-1\}$  be the smallest integer such that*

$$\left( \bigcap_{j=0}^{k_0} \operatorname{Ker} [\operatorname{Re} F(\operatorname{Im} F)^j] \right) \cap \mathbf{R}^{2n} = \{0\}.$$

*Then there exists  $C > 0$  such that for all  $u \in \mathcal{S}(\mathbf{R}^n)$ , we have*

$$\| \langle (x, D_x) \rangle^{2/(2k_0+1)} u \|_{L^2(\mathbf{R}^n)} \leq C (\| q^w(x, D_x)u \|_{L^2(\mathbf{R}^n)} + \| u \|_{L^2(\mathbf{R}^n)}). \quad (1.11)$$

The global subelliptic estimate (1.11) has turned out to be crucial in [16], in particular, when deriving sharp resolvent estimates for  $q^w(x, D_x)$  in suitable parabolic neighborhoods of the imaginary axis and when showing the exponential rate of convergence to equilibrium for the associated semigroup. The proof of Theorem 1.1 given in [18] is quite technical and is based on a delicate construction of a bounded real multiplier  $G \in C^\infty(\mathbf{R}^{2n}) \cap L^\infty(\mathbf{R}^{2n})$  such that

$$\operatorname{Re} q(X) + \frac{1}{C} H_{\operatorname{Im} q} G(X) + 1 \geq \frac{1}{C} \langle X \rangle^{\frac{2}{2k_0+1}}, \quad X \in \mathbf{R}^{2n}.$$

Here  $\langle X \rangle = (1 + |X|^2)^{1/2}$ . We remark that the existence of the multiplier  $G$  has been a key ingredient in the proof of subelliptic estimates for some classes of non-selfadjoint  $h$ -pseudodifferential operators with double characteristics in [9].

In this paper, we shall give a new and simple proof of Theorem 1.1, based on the study of the heat semigroup generated by the quadratic differential operator  $q^w = q^w(x, D_x)$ , viewed as a Fourier integral operator with a quadratic complex phase. Indeed, rather than studying the semigroup directly, following [7], [8], [23], we first perform a metaplectic FBI transform and consider the semigroup generated by the holomorphic quadratic operator  $Q$ , obtained by conjugating  $q^w(x, D_x)$  by the inverse of the FBI transform. We may then view the operator  $e^{-tQ}$  as a quadratic Fourier integral operator in the complex domain [19], [2], which is bounded between weighted spaces of holomorphic functions,

$$e^{-tQ} : H_{\Phi_0}(\mathbf{C}^n) \rightarrow H_{\Phi_t}(\mathbf{C}^n), \quad 0 \leq t \leq t_0 \ll 1.$$

Here  $\Phi_0$  is a strictly plurisubharmonic quadratic form on  $\mathbf{C}^n$  and

$$H_{\Phi_0}(\mathbf{C}^n) = \text{Hol}(\mathbf{C}^n) \cap L^2(\mathbf{C}^n; e^{-2\Phi_0} L(dx))$$

is the image of  $L^2(\mathbf{R}^n)$  under the FBI transform, with  $L(dx)$  standing for the Lebesgue measure on  $\mathbf{C}^n$ . The space  $H_{\Phi_t}(\mathbf{C}^n)$  is defined similarly, for the quadratic form  $\Phi_t \leq \Phi_0$ , whose time evolution is governed by a real Hamilton-Jacobi equation — see the discussion in Section 2. In [8] we established, as a consequence of (1.7), that for all  $t > 0$  small enough, we have the strict inequality  $\Phi_t < \Phi_0$  on  $\mathbf{C}^n \setminus \{0\}$ . Here, crucially, we are able to sharpen this bound and to show that the assumption (1.7) actually implies that

$$\Phi_t(x) \leq \Phi_0(x) - \frac{t^{2k_0+1}}{C} |x|^2, \quad x \in \mathbf{C}^n, \quad (1.12)$$

for all  $t \geq 0$  small enough. The estimate (1.12) is established here very much following the techniques of [23], developed when studying subelliptic resolvent estimates for non-selfadjoint  $h$ -pseudodifferential operators of principal type. In particular, rather than working directly with the Hamilton-Jacobi equation for  $\Phi_t$ , as was done in [8], following [23], we apply the inverse of the canonical transformation associated to the FBI transform, to replace the Hamilton-Jacobi equation for  $\Phi_t$  by another, closely related, one, which becomes easier to handle. Let us also mention that for the quadratic Kramers-Fokker-Planck operator, where we have  $k_0 = 1$ , the estimate (1.12) has been proved in [7, Section 11].

With the estimate (1.12) available, the proof of Theorem 1.1 may be completed by writing

$$(Q - z)^{-1} = \int_0^\infty e^{-t(Q-z)} dt, \quad \operatorname{Re} z < 0, \quad (1.13)$$

and observing that the smoothing properties for the resolvent  $(Q - z)^{-1}$ , equivalent to (1.11), may be derived from (1.12), via (1.13), essentially by carrying out the  $t$ -integration.

It turns out that the new method of proof of Theorem 1.1 leads also to some accurate smoothing estimates for the contraction semigroup  $e^{-tq^w}$ ,  $t \geq 0$ , in the limit of small times. Specifically, let us recall from [8, Theorem 1.2.1] that under the assumptions (1.2) and (1.7), we have for any  $t > 0$ ,

$$e^{-tq^w} : L^2(\mathbf{R}^n) \rightarrow \mathcal{S}(\mathbf{R}^n).$$

Here  $\mathcal{S}(\mathbf{R}^n)$  is the Schwartz space. The behavior of the Schwartz seminorms of  $e^{-tq^w}u$ , for  $u \in L^2(\mathbf{R}^n)$ , as  $t \rightarrow 0^+$ , is given in the following result.

**Theorem 1.2** *Let  $q : \mathbf{R}_x^n \times \mathbf{R}_\xi^n \rightarrow \mathbf{C}$  be a quadratic form with  $\operatorname{Re} q \geq 0$ , such that (1.7) holds and let us define the integer  $k_0 \in \{0, \dots, 2n-1\}$  as in Theorem 1.1. There exist  $C > 0$  and  $t_0 > 0$  such that for all  $t \in (0, t_0]$ , and all  $N \in \mathbf{N}$ , we have*

$$\| (D_x^2 + x^2)^N e^{-tq^w} \|_{\mathcal{L}(L^2(\mathbf{R}^n), L^2(\mathbf{R}^n))} \leq \frac{C^{N+1} N!}{t^{(2k_0+1)N}}. \quad (1.14)$$

Furthermore, there exist  $C > 0$  and  $t_0 > 0$  such that for all  $u \in L^2(\mathbf{R}^n)$ , all  $\mu, \nu \in \mathbf{N}^n$ , and all  $t \in (0, t_0]$ , we have

$$\| x^\mu \partial_x^\nu e^{-tq^w} u \|_{L^\infty(\mathbf{R}^n)} \leq \frac{C^{1+|\mu|+|\nu|} (\mu!)^{1/2} (\nu!)^{1/2}}{t^{\frac{(2k_0+1)}{2}(|\mu|+|\nu|+2n+s)}} \| u \|_{L^2(\mathbf{R}^n)}. \quad (1.15)$$

Here  $s > n/2$  is a fixed integer.

Theorem 1.2 implies that for any  $t > 0$  and  $u \in L^2(\mathbf{R}^n)$ , the function  $e^{-tq^w}u$  belongs to the Gelfand-Shilov space  $S_{1/2}^{1/2}(\mathbf{R}^n)$ , with the precise control of the Gelfand-Shilov seminorms, as  $t \rightarrow 0^+$ , described by the bounds (1.15). Here, following [14], we recall that a function  $f \in C^\infty(\mathbf{R}^n)$  belongs to the Gelfand-Shilov space  $S_q^p(\mathbf{R}^n)$ , with  $p, q > 0$ ,  $p+q \geq 1$ , if there exists a constant  $C \geq 1$  such that for all  $\mu, \nu \in \mathbf{N}^n$ , we have

$$\| x^\mu \partial_x^\nu f \|_{L^\infty(\mathbf{R}^n)} \leq C^{1+|\mu|+|\nu|} (\mu!)^q (\nu!)^p.$$

We refer to [14] and the references given there for a detailed discussion of the Gelfand-Shilov regularity theory. In the work [10], prepared simultaneously with the present one, using direct methods, we carry out a more detailed study of the smoothing properties of the semigroup  $e^{-tq^w}$  in the small time limit, depending on phase space directions. Comparing Theorem 1.2 with Corollary 1.2 in [10], we observe that the former result is sharper, since it provides an  $\mathcal{O}(t^{-\frac{(2k_0+1)}{2}(|\mu|+|\nu|+2n+s)})$  control for the Gelfand-Shilov seminorms of  $e^{-tq^w}u$  in the space  $S_{1/2}^{1/2}(\mathbf{R}^n)$ , whereas Corollary 1.2 in [10] gives a control in  $t^{-\frac{(2k_0+1)}{2}(|\mu|+|\nu|+s)}$  for the Gelfand-Shilov seminorms in the space  $S_{\frac{2k_0+1}{2}}^{\frac{2k_0+1}{2}}(\mathbf{R}^n)$ .

The plan of the paper is as follows. In Section 2, we study the semigroup generated by  $q^w$  on the FBI transform side and establish the estimate (1.12). Representing the resolvent of  $q^w$  as the Laplace transform of the semigroup and making use of some further direct arguments on the FBI transform side, we complete the proof of Theorem 1.1 in Section 3. In Section 4, we establish Theorem 1.2, relying upon (1.12), essentially by comparing the semigroup generated by  $q^w$  with that of the harmonic oscillator.

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## 2 The heat semigroup on the FBI transform side

We shall view  $q^w(x, D_x)$  as a closed densely defined operator on  $L^2(\mathbf{R}^n)$ , equipped with the domain

$$\mathcal{D}(q^w) = \{u \in L^2(\mathbf{R}^n); q^w(x, D_x)u \in L^2(\mathbf{R}^n)\}, \quad (2.1)$$

and let us recall from [13, Section 4] that the operator  $q^w(x, D_x)$  agrees with the graph closure of its restriction to  $\mathcal{S}(\mathbf{R}^n)$ ,

$$q^w(x, D_x) : \mathcal{S}(\mathbf{R}^n) \rightarrow \mathcal{S}(\mathbf{R}^n).$$

It follows, as observed in [13, Section 4], that the operator  $q^w(x, D_x)$  is maximal accretive and generates, in view of the Hille-Yosida theorem, a strongly continuous

contraction semigroup

$$e^{-tq^w} : L^2(\mathbf{R}^n) \rightarrow L^2(\mathbf{R}^n), \quad t \geq 0. \quad (2.2)$$

In our proof of Theorem 1.1, following [7], [8], [23], we shall be concerned with the small time behavior of the semigroup (2.2) on the FBI transform side. Let

$$Tu(x) = C \int e^{i\varphi(x,y)} u(y) dy, \quad x \in \mathbf{C}^n, \quad C > 0, \quad (2.3)$$

be a metaplectic FBI-Bargmann transform, see [22], [11]. Here  $\varphi$  is a holomorphic quadratic form on  $\mathbf{C}_x^n \times \mathbf{C}_y^n$ , such that

$$\det \varphi''_{xy} \neq 0, \quad \text{Im } \varphi''_{yy} > 0.$$

Associated to  $T$  there is a complex linear canonical transformation

$$\kappa_T : \mathbf{C}^{2n} \ni (y, -\varphi'_y(x, y)) \mapsto (x, \varphi'_x(x, y)) \in \mathbf{C}^{2n}. \quad (2.4)$$

We recall from [22, Proposition 1.1], [11, Theorem 1.3.3] that if  $C > 0$  is suitably chosen in (2.3), then  $T$  is unitary,

$$T : L^2(\mathbf{R}^n) \rightarrow H_{\Phi_0}(\mathbf{C}^n), \quad (2.5)$$

where

$$H_{\Phi_0}(\mathbf{C}^n) = \text{Hol}(\mathbf{C}^n) \cap L^2(\mathbf{C}^n, e^{-2\Phi_0(x)} L(dx)), \quad (2.6)$$

with

$$\Phi_0(x) = \sup_{y \in \mathbf{R}^n} (-\text{Im } \varphi(x, y)) \quad (2.7)$$

and  $L(dx)$  being the Lebesgue measure on  $\mathbf{C}^n$ . Let us also recall from [22, Section 1], [11, Proposition 1.3.2] that the real quadratic form  $\Phi_0$  in (2.7) is strictly plurisubharmonic on  $\mathbf{C}^n$ .

We have the exact Egorov property, [22, Proposition 1.4], [11, Theorem 1.4.2],

$$Tq^w(y, D_y)u = \tilde{q}^w(x, D_x)Tu, \quad u \in \mathcal{S}(\mathbf{R}^n), \quad (2.8)$$

where  $\tilde{q}$  is a quadratic form on  $\mathbf{C}^{2n}$  given by

$$\tilde{q} = q \circ \kappa_T^{-1}. \quad (2.9)$$



We refer to [22, Section 1], [11, Section 1.4] for a discussion of the Weyl quantization in quadratic  $H_\Phi$ -spaces. Let us also recall from [22, Section 1], [11, Proposition 1.3.2] that the canonical transformation  $\kappa_T$  maps  $\mathbf{R}^{2n}$  bijectively onto

$$\Lambda_{\Phi_0} = \left\{ \left( x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x}(x) \right); x \in \mathbf{C}^n \right\}. \quad (2.10)$$

Here the real linear subspace  $\Lambda_{\Phi_0} \subset \mathbf{C}^{2n}$  is I-Lagrangian and R-symplectic, and in particular, it is maximally totally real. The holomorphic quadratic form  $\tilde{q}$  is therefore uniquely determined by its restriction to  $\Lambda_{\Phi_0}$ , and we may notice, in view of (1.2) and (2.9), that

$$\operatorname{Re} \tilde{q} \left( x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x}(x) \right) \geq 0, \quad x \in \mathbf{C}^n. \quad (2.11)$$

Let us simplify the notation and write in what follows,  $Q = \tilde{q}^w(x, D_x)$ . The operator  $Q$  is a holomorphic quadratic differential operator and we would like to study the unbounded operator  $e^{tQ}$  on  $H_{\Phi_0}(\mathbf{C}^n)$ , for  $0 \leq t \leq t_0$ , with  $t_0 > 0$  sufficiently small, see also [1]. To that end, let us consider the evolution problem

$$(\partial_t - Q)u(t, x) = 0, \quad u|_{t=0} = u_0 \in H_{\Phi_0}(\mathbf{C}^n), \quad (2.12)$$

for  $t \in [0, t_0]$ , which we can solve by a geometric optics construction. Let  $\phi(t, x, \eta)$  be the holomorphic quadratic form on  $\mathbf{C}_x^n \times \mathbf{C}_\eta^n$ , depending smoothly on  $t \in [0, t_0]$ , with  $t_0 > 0$  sufficiently small, and satisfying the Hamilton-Jacobi equation,

$$i\partial_t \phi(t, x, \eta) - \tilde{q}(x, \partial_x \phi(t, x, \eta)) = 0, \quad \phi(0, x, \eta) = x \cdot \eta. \quad (2.13)$$

From the general Hamilton-Jacobi theory [3, Chapter 1], we know that for  $t \in [0, t_0]$ , with  $t_0 > 0$  small enough, the quadratic form  $\phi(t, x, \eta)$  can be obtained as a generating function for the complex linear canonical transformation

$$\exp(itH_{\tilde{q}}) : \mathbf{C}^{2n} \ni (\partial_\eta \phi(t, x, \eta), \eta) \mapsto (x, \partial_x \phi(t, x, \eta)) \in \mathbf{C}^{2n}. \quad (2.14)$$

Here, when  $f$  is a holomorphic function on  $\mathbf{C}^{2n} = \mathbf{C}_x^n \times \mathbf{C}_\xi^n$ , the Hamilton vector field  $H_f$  of  $f$  is a holomorphic vector field given by the usual formula,

$$H_f = \sum_{j=1}^n \left( \frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial \xi_j} \right).$$

It follows that for  $0 \leq t \leq t_0 \ll 1$ , the solution operator  $e^{tQ}$  to (2.12) is given by the following quadratic Fourier integral operator in the complex domain,

$$e^{tQ}u(x) = \frac{1}{(2\pi)^n} \iint_{\Gamma(x,t)} e^{i(\phi(t,x,\eta) - y \cdot \eta)} a(t) u(y) dy d\eta, \quad u \in H_{\Phi_0}(\mathbf{C}^n). \quad (2.15)$$

Here the amplitude  $a(t)$ , depending smoothly on  $t$ , with  $a(0) = 1$ , is obtained by solving a suitable transport equation, which we need not specify here. When explaining the choice of the contour of integration  $\Gamma(x,t)$  in (2.15), we shall follow the discussion in Appendix B of [2], which in turn can be viewed as a linear version of the general theory described in [19, Chapters 3,4]. Let  $\Phi_t$  be a real strictly plurisubharmonic quadratic form on  $\mathbf{C}^n$ , depending smoothly on  $t \in [0, t_0]$ , for  $t_0 > 0$  small enough, such that if we set

$$\Lambda_{\Phi_t} = \left\{ \left( x, \frac{2}{i} \frac{\partial \Phi_t}{\partial x}(x) \right); x \in \mathbf{C}^n \right\} \quad (2.16)$$

then

$$\Lambda_{\Phi_t} = \exp \left( t \widehat{H_{i\bar{q}}} \right) (\Lambda_{\Phi_0}). \quad (2.17)$$

Here, when  $\nu$  is a vector field of type  $(1,0)$  on  $\mathbf{C}^{2n}$ , we let  $\widehat{\nu} = \nu + \bar{\nu}$  be the corresponding real vector field. An application of [2, Proposition B.1] and [2, Proposition B.2] allows us to conclude that the plurisubharmonic quadratic form

$$\mathbf{C}^n \times \mathbf{C}^n \ni (y, \eta) \mapsto -\text{Im} (\phi(t, x, \eta) - y \cdot \eta) + \Phi_0(y)$$

has a unique critical point  $(y_c(x, t), \eta_c(x, t))$  for each  $x \in \mathbf{C}^n$  and  $t \in [0, t_0]$ , which is non-degenerate of signature  $(2n, 2n)$ . Furthermore, we have

$$\Phi_t(x) = \text{vc}_{y,\eta} (-\text{Im} (\phi(t, x, \eta) - y \cdot \eta) + \Phi_0(y)),$$

where the general notation  $\text{vc}_{y,\eta}(\dots)$  stands for the critical value with respect to  $y$  and  $\eta$  of  $(\dots)$ . The contour  $\Gamma(x, t) \subset \mathbf{C}_{y,\eta}^{2n}$  in (2.15) is a so called good contour [19, Chapter 3], which is an affine subspace of  $\mathbf{C}^{2n}$  of real dimension  $2n$ , passing through the critical point  $(y_c(x, t), \eta_c(x, t))$ , and along which we have

$$-\text{Im} (\phi(t, x, \eta) - y \cdot \eta) + \Phi_0(y) - \Phi_t(x) \asymp -(|y - y_c(x, t)|^2 + |\eta - \eta_c(x, t)|^2).$$

Applying [2, Proposition B.3], we conclude that we have a bounded operator

$$e^{tQ} : H_{\Phi_0}(\mathbf{C}^n) \rightarrow H_{\Phi_t}(\mathbf{C}^n), \quad 0 \leq t \leq t_0 \ll 1, \quad (2.18)$$

where similarly to (2.6), we set

$$H_{\Phi_t}(\mathbf{C}^n) = \text{Hol}(\mathbf{C}^n) \cap L^2(\mathbf{C}^n, e^{-2\Phi_t(x)} L(dx)).$$

Associated to the function  $\Phi(t, x) = \Phi_t(x)$  is the manifold

$$\tau = \frac{\partial \Phi}{\partial t}, \quad \xi = \frac{2}{i} \frac{\partial \Phi}{\partial x},$$

in  $\mathbf{R}_{t,\tau}^2 \times \mathbf{C}_{x,\xi}^{2n}$ , which is Lagrangian with respect to the real symplectic form

$$d\tau \wedge dt - \text{Im } \sigma, \quad (2.19)$$

where

$$\sigma = \sum_{j=1}^n d\xi_j \wedge dx_j,$$

is the complex symplectic  $(2, 0)$ -form on  $\mathbf{C}^{2n} = \mathbf{C}_x^n \times \mathbf{C}_\xi^n$ . Let us also recall the general relation [8],

$$\widehat{H}_{i\tilde{q}} = H_{-\text{Re } \tilde{q}}^{-\text{Im } \sigma}, \quad (2.20)$$

where  $H_g^{-\text{Im } \sigma}$  is the Hamilton vector field of a function  $g \in C^1(\mathbf{C}^{2n}, \mathbf{R})$ , computed with respect to the real symplectic form  $-\text{Im } \sigma$ . The Hamilton-Jacobi theory applied with respect to the real symplectic form in (2.19) tells us therefore that the function  $\Phi(t, x)$  satisfies the real Hamilton-Jacobi equation,

$$\frac{\partial \Phi}{\partial t}(t, x) - \text{Re } \tilde{q} \left( x, \frac{2}{i} \frac{\partial \Phi}{\partial x}(t, x) \right) = 0, \quad \Phi(0, \cdot) = \Phi_0, \quad (2.21)$$

for  $x \in \mathbf{C}^n$ ,  $0 \leq t \leq t_0 \ll 1$ . See also [20, Section 3] and [8, Section 3].

Now (2.20) implies that the function  $\text{Re } \tilde{q}$  is constant along the flow of the Hamilton vector field  $\widehat{H}_{i\tilde{q}}$ , and it follows from (2.11) and (2.17) that  $\text{Re } \tilde{q}|_{\Lambda_{\Phi_t}} \geq 0$ . Using (2.21) we conclude that

$$\frac{\partial \Phi}{\partial t}(t, x) \geq 0,$$

so that the function  $t \mapsto \Phi_t(x)$  is increasing.

*Remark.* Let us consider estimates for the operator norm of

$$e^{tQ} \in \mathcal{L}(H_{\Phi_0}(\mathbf{C}^n), H_{\Phi_t}(\mathbf{C}^n)),$$

for  $0 \leq t \leq t_0$ . When doing so, let us set

$$L_{\Phi_0}^2(\mathbf{C}^n) = L^2(\mathbf{C}^n; e^{-2\Phi_0} L(dx)),$$

and let  $u \in H_{\Phi_0}(\mathbf{C}^n)$  be such that

$$\langle x \rangle^N u \in L_{\Phi_0}^2(\mathbf{C}^n), \quad (2.22)$$

for all  $N \in \mathbf{N}$ . We recall from [22] that functions in  $H_{\Phi_0}(\mathbf{C}^n)$  satisfying (2.22) are precisely those for which  $T^{-1}u \in \mathcal{S}(\mathbf{R}^n)$ , and in particular, such functions are dense in  $H_{\Phi_0}(\mathbf{C}^n)$ . Let us differentiate the scalar product

$$(e^{tQ}u, e^{tQ}u)_{H_{\Phi_t}} = (u(t), u(t))_{H_{\Phi_t}}$$

with respect to  $t$ . We get

$$\begin{aligned} \frac{d}{dt}(u(t), u(t))_{H_{\Phi_t}} = \\ (Qu(t), u(t))_{H_{\Phi_t}} + (u(t), Qu(t))_{H_{\Phi_t}} - 2 \int |u(t)|^2 e^{-2\Phi_t(x)} \frac{\partial \Phi_t}{\partial t}(x) L(dx). \end{aligned}$$

Here the first two terms in the right hand side can be simplified by means of the quantization-multiplication formula [21, Theorem 1.2], [11, Proposition 1.4.4], which becomes exact in the present quadratic case,

$$(Qu(t), u(t))_{H_{\Phi_t}} = \int \tilde{q} \left( x, \frac{2}{i} \frac{\partial \Phi_t}{\partial x}(x) \right) |u(t)|^2 e^{-2\Phi_t(x)} L(dx) + b(t) \|u(t)\|_{H_{\Phi_t}}^2, \quad (2.23)$$

where  $b \in C^\infty([0, t_0])$ . We refer to [1, Section 2] for an explicit computation of  $\Phi_t$  and  $b(t)$ , in a particular FBI representation. Combining (2.23) with its analog for  $(u(t), Qu(t))_{H_{\Phi_t}}$  and using (2.21) we obtain that

$$\frac{d}{dt} \|u(t)\|_{H_{\Phi_t}}^2 = (2\operatorname{Re} b(t)) \|u(t)\|_{H_{\Phi_t}}^2.$$

Setting  $B(t) = \int_0^t \operatorname{Re} b(s) ds \in C^\infty([0, t_0])$ , we conclude that the operator

$$e^{-B(t)} e^{tQ} : H_{\Phi_0}(\mathbf{C}^n) \rightarrow H_{\Phi_t}(\mathbf{C}^n), \quad 0 \leq t \leq t_0, \quad (2.24)$$

is an isometry. As we shall discuss in the next section, it follows from the general theory [19, Chapters 3,4] that it is necessarily a bijection, and therefore the map (2.24) is unitary.

Let us recall that the integer  $0 \leq k_0 \leq 2n - 1$  has been introduced in (1.8). The following is the main result of this section.

**Theorem 2.1** *There exist  $t_0 > 0$  and  $C > 0$  such that for all  $t \in [0, t_0]$  we have*

$$\Phi_t(x) \geq \Phi_0(x) + \frac{t^{2k_0+1}}{C} |x|^2, \quad x \in \mathbf{C}^n. \quad (2.25)$$

When proving Theorem 2.1, we shall follow the arguments of [23] closely, and indeed, the following discussion can be viewed as a straightforward adaptation of the analysis of [23, Sections 2,3] to the quadratic case. Let us introduce a  $C^\infty$ -family of linear I-Lagrangian R-symplectic manifolds  $\Lambda_t$ , given by

$$\Lambda_t = \kappa_T^{-1}(\Lambda_{\Phi_t}) \subset \mathbf{C}^{2n}, \quad t \in [0, t_0], \quad 0 < t_0 \ll 1. \quad (2.26)$$

Using (2.9) and (2.17), we can write

$$\Lambda_t = \kappa_T^{-1} \circ \exp(itH_{\tilde{q}})(\Lambda_{\Phi_0}) = \kappa_T^{-1} \circ \exp(itH_{\tilde{q}}) \circ \kappa_T(\mathbf{R}^{2n}) = \exp(itH_q)(\mathbf{R}^{2n}).$$

Here we have identified the flow  $\exp(t\widehat{H}_{i\tilde{q}})$  of the real vector field  $\widehat{H}_{i\tilde{q}}$  with the holomorphic flow of the holomorphic vector field  $H_{i\tilde{q}}$ , restricted to small positive  $t \in \mathbf{R}$ . The I-Lagrangian manifold  $\Lambda_t \subset \mathbf{C}_{x,\xi}^{2n}$  is an  $\mathcal{O}(t)$ -perturbation of  $\mathbf{R}^{2n}$  in the sense of linear subspaces, and the real 1-form  $\text{Im}(\xi \cdot dx)$  is closed, and hence exact, on  $\Lambda_t$ . It follows that there exists a unique real-valued quadratic form  $G_t$  on  $\mathbf{R}^{2n}$ , depending smoothly on  $t \in \text{neigh}(0, [0, \infty))$ , such that  $G_0 = 0$  and

$$\Lambda_t = \kappa_T^{-1}(\Lambda_{\Phi_t}) = \{X + iH_{G_t}(X); X \in \mathbf{R}^{2n}\}. \quad (2.27)$$

Here  $H_{G_t}$  is the Hamilton vector field of  $G_t$ .

*Example.* Let  $q : \mathbf{R}_x^n \times \mathbf{R}_\xi^n \rightarrow \mathbf{R}$  be real-valued and positive semi-definite. Then, using the fact that  $H_q = 2F$  is real, we write for small  $t \in \mathbf{R}$ ,

$$\Lambda_t = \exp(itH_q)(\mathbf{R}^{2n}) = \{X + i \tan(2tF)X; X \in \mathbf{R}^{2n}\},$$

where  $\tan(2tF) = \sin(2tF)(\cos(2tF))^{-1}$ . Observing that  $\cos F$  is symmetric and  $\sin F$  is skew-symmetric with respect to the symplectic form  $\sigma$ , we see that the bilinear form

$$\mathbf{R}^{2n} \times \mathbf{R}^{2n} \ni (X, Y) \mapsto \sigma(X, \tan(2tF)Y)$$

is symmetric. It follows that the quadratic form  $G_t$  in (2.27) is given by

$$G_t(X) = \frac{1}{2} \sigma(X, \tan(2tF)X).$$

In [23, Proposition 2.1], it is explained how to recover the quadratic form  $\Phi_t$  in (2.16) from  $G_t$ , and we recall from this result that

$$\Phi_t(x) = \text{vc}_{(y,\eta) \in \mathbf{C}^n \times \mathbf{R}^n} (-\text{Im } \varphi(x, y) - \eta \cdot \text{Im } y + G_t(\text{Re } y, \eta)). \quad (2.28)$$

Here  $\varphi(x, y)$  is the phase of the FBI-Bargmann transform in (2.3). For the convenience of the reader, we shall now discuss briefly the derivation of the formula (2.28), following [23]. When doing so, let us observe that the point  $(y, \eta) \in \mathbf{C}^n \times \mathbf{R}^n$  is a critical point of the function

$$\mathbf{C}^n \times \mathbf{R}^n \ni (y, \eta) \mapsto -\text{Im } \varphi(x, y) - \eta \cdot \text{Im } y + G_t(\text{Re } y, \eta)$$

precisely when we have

$$\text{Im } y = \nabla_\eta G_t(\text{Re } y, \eta), \quad \frac{\partial}{\partial \text{Re } y} \text{Im } \varphi(x, y) = \nabla_y G_t(\text{Re } y, \eta), \quad (2.29)$$

and

$$\frac{\partial}{\partial \text{Im } y} \text{Im } \varphi(x, y) + \eta = 0. \quad (2.30)$$

In view of the Cauchy-Riemann equations, we can write

$$\frac{\partial}{\partial \text{Re } y} \text{Im } \varphi(x, y) = \text{Im } \varphi'_y(x, y), \quad \frac{\partial}{\partial \text{Im } y} \text{Im } \varphi(x, y) = \text{Re } \varphi'_y(x, y),$$

and therefore, (2.29), (2.30) can equivalently be stated as follows,

$$(y, -\varphi'_y(x, y)) = (\text{Re } y, \eta) + iH_{G_t}(\text{Re } y, \eta).$$

We conclude that the critical value in (2.28) is attained at a unique critical point  $(y, \eta) = (y(x, t), \eta(x, t)) \in \mathbf{C}^n \times \mathbf{R}^n$ , which is determined by the condition that the linear canonical transformation  $\kappa_T$  in (2.4) maps the point

$$(\text{Re } y, \eta) + iH_{G_t}(\text{Re } y, \eta) \in \Lambda_t = \kappa_T^{-1}(\Lambda_{\Phi_t}) \quad (2.31)$$

to the point  $(x, \varphi'_x(x, y))$ , situated above  $x \in \mathbf{C}^n$ . As verified in [23], the critical point is non-degenerate, and in order to complete the proof of (2.28), it suffices to observe that if  $\Phi_t(x)$  stands for the critical value in (2.28), we have, using that  $\Phi_t$  is a critical value,

$$\frac{2}{i} \frac{\partial \Phi_t}{\partial x}(x) = \frac{2}{i} \frac{\partial}{\partial x} (-\text{Im } \varphi(x, y)) = \varphi'_x(x, y),$$

where  $(y, \eta)$  is the corresponding critical point.

Continuing to follow [23], from [23, Proposition 2.4], let us also recall the following inversion formula for  $(y, \eta) \in \mathbf{R}^{2n}$ ,

$$G_t(y, \eta) = \text{vc}_{(x, \theta) \in \mathbf{C}^n \times \mathbf{R}^n} (\text{Im } \varphi(x, y + i\theta) + \eta \cdot \theta + \Phi_t(x)). \quad (2.32)$$

Using that  $\Phi_t \geq \Phi_0$  we conclude as in [23, Section 2], that  $G_t \geq 0$  for all  $t \in [0, t_0]$ , for  $t_0 > 0$  sufficiently small.

We shall next show, following [23, Section 3], that the real Hamilton-Jacobi equation (2.21) for  $\Phi_t$  implies a similar equation for  $G_t$ . To this end, let  $(x(t, y, \eta), \theta(t, y, \eta))$  be the critical point in (2.32), and let us write, using the fact that  $G_t$  is the critical value in (2.32) together with (2.21),

$$\frac{\partial G_t}{\partial t}(y, \eta) = \frac{\partial \Phi_t}{\partial t}(x(t, y, \eta)) = \text{Re } \tilde{q} \left( x, \frac{2}{i} \frac{\partial \Phi_t}{\partial x}(x) \right) \Big|_{x=x(t, y, \eta)}.$$

Now as discussed above, the critical points in (2.28) and (2.32) are related to  $\kappa_T$ , so that

$$\left( x(t, y, \eta), \frac{2}{i} \frac{\partial \Phi_t}{\partial x}(x(t, y, \eta)) \right) = \kappa_T((y, \eta) + iH_{G_t}(y, \eta)).$$

We obtain therefore the following equation for the smooth family of quadratic forms  $G_t$ ,

$$\frac{\partial G_t}{\partial t}(y, \eta) = \text{Re } (q((y, \eta) + iH_{G_t}(y, \eta))), \quad (2.33)$$

which we state as an initial value problem, for  $X \in \mathbf{R}^{2n}$  and  $t \in [0, t_0]$ , for  $t_0 > 0$  small enough,

$$\frac{\partial G_t}{\partial t}(X) - \text{Re } (q(X + iH_{G_t}(X))) = 0, \quad G_0 = 0. \quad (2.34)$$

Using the fact that  $q$  is quadratic, we write

$$\text{Re } (q(X + iH_{G_t}(X))) = \text{Re } q(X) + H_{\text{Im } q} G_t(X) - \text{Re } q(H_{G_t}(X)), \quad (2.35)$$

and the equation (2.34) becomes

$$\frac{\partial G_t}{\partial t}(X) - H_{\text{Im } q} G_t(X) + \text{Re } q(H_{G_t}(X)) = \text{Re } q(X), \quad G_0 = 0. \quad (2.36)$$

Here, as we have already observed, the quadratic form  $G_t \geq 0$  is nonnegative, and therefore, for some constant  $C > 0$ , we have for all  $t \in [0, t_0]$ ,

$$0 \leq \operatorname{Re} q(H_{G_t}(X)) \leq \mathcal{O}(1) |\nabla G_t(X)|^2 \leq CG_t(X). \quad (2.37)$$

We get, using (2.36) and (2.37),

$$\frac{\partial G_t}{\partial t}(X) - H_{\operatorname{Im} q} G_t(X) + CG_t(X) \geq \operatorname{Re} q(X), \quad G_0 = 0. \quad (2.38)$$

Considering (2.38) as a differential inequality along the integral curves of  $H_{-\operatorname{Im} q}$  and using Gronwall's lemma we see that for all  $t \in [0, t_0]$  and  $X \in \mathbf{R}^{2n}$ , we have for some  $C > 0$ ,

$$G_t(\exp(tH_{-\operatorname{Im} q})(X)) \geq \frac{1}{C} \int_0^t \operatorname{Re} q(\exp(sH_{-\operatorname{Im} q})(X)) ds. \quad (2.39)$$

Let us put

$$J(t, X) = \int_0^t \operatorname{Re} q(\exp(sH_{\operatorname{Im} q})(X)) ds, \quad t \in [0, t_0], \quad X \in \mathbf{R}^{2n}, \quad (2.40)$$

so that  $0 \leq J(t, X)$  is a quadratic form on  $\mathbf{R}^{2n}$  with  $C^\infty$  dependence on  $t \in [0, t_0]$ . From [8, Proposition 2.0.1], let us recall that for each  $t > 0$ , the quadratic form  $X \mapsto J(t, X)$  is positive definite on  $\mathbf{R}^{2n}$ . The following result, which is an analog of [23, Proposition 3.2], is a sharpening of this basic observation.

**Proposition 2.2** *There exist  $t_0 > 0$  and  $C > 0$  such that for all  $t \in [0, t_0]$  we have*

$$J(t, X) \geq \frac{t^{2k_0+1}}{C} |X|^2, \quad X \in \mathbf{R}^{2n}. \quad (2.41)$$

*Proof:* We proceed similarly to the proof of Proposition 3.2 in [23]. Let  $X \in \mathbf{R}^{2n}$  be such that  $|X| = 1$  and let  $j \in \mathbf{N}$ ,  $0 \leq j \leq k_0$ , be such that

$$\operatorname{Re} q(\exp(tH_{\operatorname{Im} q})(X)) = at^{2j} + \mathcal{O}(t^{2j+1}), \quad t \rightarrow 0, \quad (2.42)$$

where  $a > 0$ . Here we use the property (1.9). We claim that there exist  $C_X > 0$ ,  $t_X \in (0, 1)$ , and a neighborhood  $V_X$  of  $X$  in  $S^{n-1}$  such that for all  $t \in (0, t_X]$  and all  $Y \in V_X$ , we have

$$J(t, Y) \geq \frac{t^{2j+1}}{C_X}. \quad (2.43)$$



Seeking a contradiction, let us assume that (2.43) does not hold. Then there exist sequences  $0 < t_\nu \rightarrow 0$ ,  $Y_\nu \in S^{n-1}$ ,  $Y_\nu \rightarrow X$  such that

$$\frac{J(t_\nu, Y_\nu)}{t_\nu^{2j+1}} \rightarrow 0, \quad \nu \rightarrow \infty. \quad (2.44)$$

Using the fact that the function  $t \mapsto J(t, X)$  is increasing, we get from (2.44),

$$\sup_{0 \leq t \leq t_\nu} \frac{J(t, Y_\nu)}{t_\nu^{2j+1}} \rightarrow 0. \quad (2.45)$$

A Taylor expansion gives that

$$J(t, Y_\nu) = a_\nu^{(0)} + a_\nu^{(1)}t + \dots + a_\nu^{(2j+1)}t^{2j+1} + \mathcal{O}(t^{2j+2}), \quad t \rightarrow 0, \quad (2.46)$$

and let us define

$$0 \leq u_\nu(s) = \frac{J(t_\nu s, Y_\nu)}{t_\nu^{2j+1}}, \quad 0 \leq s \leq 1,$$

so that according to (2.45), we have

$$\sup_{0 \leq s \leq 1} u_\nu(s) \rightarrow 0, \quad \nu \rightarrow \infty. \quad (2.47)$$

On the other hand, using (2.46), we write

$$u_\nu(s) = p_\nu(s) + \mathcal{O}(t_\nu s^{2j+2}), \quad (2.48)$$

where  $p_\nu$  is a polynomial of degree  $2j+1$  given by

$$p_\nu(s) = \frac{a_\nu^{(0)}}{t_\nu^{2j+1}} + \frac{a_\nu^{(1)}}{t_\nu^{2j}}s + \dots + a_\nu^{(2j+1)}s^{2j+1}.$$

It follows from (2.47) and (2.48) that  $p_\nu \rightarrow 0$  uniformly on  $[0, 1]$  as  $\nu \rightarrow \infty$ , and since all norms on a finite-dimensional vector space are equivalent, we see that the coefficients of  $p_\nu$  all tend to 0 as  $\nu \rightarrow \infty$ . In particular,

$$a_\nu^{(2j+1)} = \frac{1}{(2j+1)!} (\partial_t^{2j+1} J)(0, Y_\nu) \rightarrow 0.$$

On the other hand, (2.42) shows that

$$\partial_t^{2j+1} J(0, X) = (2j)!a > 0,$$

and this contradiction establishes the claim. In particular, we see that for all  $t \in [0, t_X]$  and all  $Y \in V_X$  we have

$$J(t, Y) \geq \frac{t^{2k_0+1}}{C_X}.$$

Covering the compact set  $S^{n-1}$  by finitely many open neighborhoods of the form  $V_{X_1}, \dots, V_{X_N}$  and letting  $C = \max_{1 \leq j \leq N} C_{X_j}$ ,  $t_0 = \min_{1 \leq j \leq N} t_{X_j} > 0$ , we conclude that for all  $Y \in S^{n-1}$ , we have

$$J(t, Y) \geq \frac{t^{2k_0+1}}{C}, \quad 0 \leq t \leq t_0.$$

Using the fact that  $Y \mapsto J(t, Y)$  is quadratic, we conclude the proof.  $\square$

Coming back to (2.39), we observe that the conclusion of Proposition 2.2 remains valid also for the function

$$\tilde{J}(t, X) = \int_0^t \operatorname{Re} q(\exp(sH_{-\operatorname{Im} q})(X)) \, ds.$$

Indeed, the Hamilton map of the quadratic form  $X \mapsto \overline{q(X)}$  is the complex conjugate  $\overline{F}$  of the Hamilton map  $F$  of  $q$ , and it follows therefore that the assumptions of Theorem 1.1 are also valid for  $\overline{q}$ , with the same value of  $k_0$ . Combining this observation with (2.39) we obtain that there exist  $C > 0$  and  $t_0 > 0$  such that the following estimate holds,

$$G_t(X) \geq \frac{t^{2k_0+1}}{C} |X|^2, \quad t \in [0, t_0], \quad X \in \mathbf{R}^{2n}. \quad (2.49)$$

We next observe that, as explained in [23, Section 2], the inequality (2.49) together with (2.28) shows that

$$\Phi_t(x) \geq \Psi_t(x), \quad x \in \mathbf{C}^n, \quad t \in [0, t_0], \quad (2.50)$$

where

$$\Psi_t(x) = \operatorname{vc}_{(y, \eta) \in \mathbf{C}^n \times \mathbf{R}^n} \left( -\operatorname{Im} \varphi(x, y) - \eta \cdot \operatorname{Im} y + \frac{t^{2k_0+1}}{C} |(\operatorname{Re} y, \eta)|^2 \right). \quad (2.51)$$

The unique critical point in (2.51) satisfies

$$y(x, t) = y_c(x) + \mathcal{O}(t^{2k_0+1} |x|), \quad \eta(x, t) = -\varphi'_y(x, y_c(x)) + \mathcal{O}(t^{2k_0+1} |x|),$$

where  $y_c(x) \in \mathbf{R}^n$  is the unique point such that

$$\Phi_0(x) = -\operatorname{Im} \varphi(x, y_c(x)).$$

Thus,  $y_c(x)$  is the unique critical point of the function

$$\mathbf{R}^n \ni y \mapsto -\operatorname{Im} \varphi(x, y),$$

so that  $\varphi'_y(x, y_c(x))$  is real. From [19, Chapter 7], [11, Section 1.3] we also recall that the map  $\mathbf{C}^n \ni x \mapsto (y_c(x), -\varphi'_y(x, y_c(x))) \in \mathbf{R}^n \times \mathbf{R}^n$  is a real linear isomorphism. It follows that

$$\begin{aligned} \Psi_t(x) &= \Phi_0(x) + \frac{t^{2k_0+1}}{C} |(y_c(x), -\varphi'_y(x, y_c(x)))|^2 + \mathcal{O}(t^{4k_0+2} |x|^2) \\ &\geq \Phi_0(x) + \frac{t^{2k_0+1}}{\mathcal{O}(1)} |x|^2, \quad 0 \leq t \leq t_0 \ll 1, \quad x \in \mathbf{C}^n. \end{aligned} \quad (2.52)$$

In view of (2.50) and (2.52), the proof of Theorem 2.1 is complete.

*Remark.* For future reference, let us notice that the arguments developed in this section apply equally well to the bounded operator

$$e^{-tQ} : H_{\Phi_0}(\mathbf{C}^n) \rightarrow H_{\Phi_0}(\mathbf{C}^n), \quad t \geq 0,$$

and allow us to conclude that for all  $t \in [0, t_0]$ , with  $t_0 > 0$  sufficiently small, the operator  $e^{-tQ}$  is bounded,

$$e^{-tQ} : H_{\Phi_0}(\mathbf{C}^n) \rightarrow H_{\tilde{\Phi}_t}(\mathbf{C}^n), \quad (2.53)$$

where  $\tilde{\Phi}_t$  is a real strictly plurisubharmonic quadratic form on  $\mathbf{C}^n$ , depending smoothly on  $t \in [0, t_0]$ , such that

$$\tilde{\Phi}_t(x) \leq \Phi_0(x) - \frac{t^{2k_0+1}}{C} |x|^2, \quad x \in \mathbf{C}^n, \quad t \in [0, t_0], \quad (2.54)$$

for some constant  $C > 0$ . This result can be viewed as a sharpening of [8, Lemma 3.1.2].

### 3 Subelliptic estimates

In this section, we shall complete the proof of Theorem 1.1, relying crucially on the estimate (2.25). When doing so, let us recall from [8, Theorem 1.2.2] that the

spectrum of the closed densely defined quadratic operator  $Q : H_{\Phi_0}(\mathbf{C}^n) \rightarrow H_{\Phi_0}(\mathbf{C}^n)$  is discrete, contained in the open right half plane  $\{z \in \mathbf{C}; \operatorname{Re} z > 0\}$ . Here the domain of  $Q$  is given by

$$\mathcal{D}(Q) = \{u \in H_{\Phi_0}(\mathbf{C}^n); Qu \in H_{\Phi_0}(\mathbf{C}^n)\}.$$

We shall be concerned with deriving estimates for the bounded operator

$$Q^{-1} : H_{\Phi_0}(\mathbf{C}^n) \rightarrow H_{\Phi_0}(\mathbf{C}^n),$$

and in doing so we write

$$Q^{-1} = \int_0^\infty e^{-tQ} dt. \quad (3.1)$$

Here

$$e^{-tQ} : H_{\Phi_0}(\mathbf{C}^n) \rightarrow H_{\Phi_0}(\mathbf{C}^n), \quad t \geq 0,$$

is the strongly continuous contraction semigroup generated by  $Q$ , and from [8, Theorem 1.2.3] we recall that the norm of  $e^{-tQ}$  on  $H_{\Phi_0}(\mathbf{C}^n)$  decays exponentially as  $t \rightarrow +\infty$ , so that the integral in (3.1) converges in  $\mathcal{L}(H_{\Phi_0}(\mathbf{C}^n), H_{\Phi_0}(\mathbf{C}^n))$ .

Let  $t_0 > 0$  be small enough fixed and observe that using the semigroup property we can write,

$$Q^{-1} = \int_0^{t_0} e^{-tQ} dt + \int_{t_0}^\infty e^{-tQ} dt = \int_0^{t_0} e^{-tQ} dt + e^{-t_0Q} Q^{-1}. \quad (3.2)$$

It follows from [8], see also (2.53), (2.54), that there exists  $\eta > 0$  such that the operator  $e^{-t_0Q}$  is bounded,

$$e^{-t_0Q} : H_{\Phi_0}(\mathbf{C}^n) \rightarrow H_{\Phi_0 - \eta|x|^2}(\mathbf{C}^n),$$

and we shall therefore begin by discussing estimates for the first term in the right hand side of (3.2), given by the operator

$$R = \int_0^{t_0} e^{-tQ} dt : H_{\Phi_0}(\mathbf{C}^n) \rightarrow H_{\Phi_0}(\mathbf{C}^n).$$

Let  $u, v$  be holomorphic functions on  $\mathbf{C}^n$ , such that

$$\langle x \rangle^N u, \quad \langle x \rangle^N v \in L_{\Phi_0}^2(\mathbf{C}^n),$$

for all  $N \in \mathbf{N}$ . Consider the scalar product

$$(Ru, v)_{H_{\Phi_0}} = \iint 1_{[0, t_0]}(t) (e^{-tQ} u(x)) \overline{v(x)} e^{-2\Phi_0(x)} L(dx) dt, \quad (3.3)$$

which can be written as follows,

$$(Ru, v)_{H_{\Phi_0}} = \int_0^{t_0} (e^{-tQ} u, v)_{H_{\Phi_0}} dt = \int_0^{t_0} (e^{-tQ/2} u, e^{-tQ^*/2} v)_{H_{\Phi_0}} dt. \quad (3.4)$$

Here we have used the semigroup property of  $e^{-tQ}$  and the fact that the adjoint semigroup of  $e^{-tQ}$  is generated by the adjoint  $Q^*$  of  $Q$  in  $H_{\Phi_0}(\mathbf{C}^n)$ , see [17, Corollary 1.10.6]. From [13, Section 4], [9, Section 3], we recall that  $Q^*$  is a closed densely defined holomorphic quadratic differential operator on  $H_{\Phi_0}(\mathbf{C}^n)$  such that

$$Q^* = T\bar{q}^w(y, D_y)T^{-1},$$

where the domain of  $\bar{q}^w(y, D_y)$  is given by  $\{u \in L^2(\mathbf{R}^n); \bar{q}^w(y, D_y)u \in L^2(\mathbf{R}^n)\}$ . Furthermore, as already observed in Section 2, the quadratic form  $X \mapsto \overline{q(X)}$  also satisfies the assumptions of Theorem 1.1, with the same value of  $k_0$ .

Using (3.4), we get

$$(Ru, v)_{H_{\Phi_0}} = \iint 1_{[0, t_0]}(t) (e^{-tQ/2} u(x)) \overline{(e^{-tQ^*/2} v(x))} e^{-2\Phi_0(x)} L(dx) dt, \quad (3.5)$$

and by the Cauchy-Schwarz inequality, we conclude that

$$|(Ru, v)_{H_{\Phi_0}}|^2$$

does not exceed the product

$$\left( \iint 1_{[0, t_0]}(t) |e^{-tQ/2} u(x)|^2 e^{-2\Phi_0(x)} L(dx) dt \right) \times \left( \iint 1_{[0, t_0]}(t) |e^{-tQ^*/2} v(x)|^2 e^{-2\Phi_0(x)} L(dx) dt \right). \quad (3.6)$$

When estimating (3.6), we observe that according to the discussion in Section 2, we have a bounded quadratic elliptic Fourier integral operator in the complex domain,

$$e^{tQ/2} : H_{\Phi_0}(\mathbf{C}^n) \rightarrow H_{\Phi_{t/2}}(\mathbf{C}^n), \quad 0 \leq t \leq t_0, \quad (3.7)$$

where the strictly plurisubharmonic quadratic form  $\Phi_t$  is given by (2.17), (2.21). An application of [2, Proposition B.4] allows us to conclude that the operator in (3.7) has a continuous two-sided inverse,

$$e^{-tQ/2} : H_{\Phi_{t/2}}(\mathbf{C}^n) \rightarrow H_{\Phi_0}(\mathbf{C}^n), \quad 0 \leq t \leq t_0, \quad (3.8)$$

with the operator norm  $\|e^{-tQ/2}\|_{\mathcal{L}(H_{\Phi_{t/2}}(\mathbf{C}^n), H_{\Phi_0}(\mathbf{C}^n))}$  uniformly bounded, for  $t \in [0, t_0]$ . Let us write, when  $t \in [0, t_0]$ , with  $t_0 > 0$  sufficiently small,

$$\int |e^{-tQ/2}u(x)|^2 e^{-2\Phi_0(x)} L(dx) \leq C(t_0) \int |u(x)|^2 e^{-2\Phi_{t/2}(x)} L(dx), \quad (3.9)$$

for some constant  $C(t_0) > 0$ . Integrating (3.9) with respect to  $t \in [0, t_0]$  and using Theorem 2.1, we conclude that the first factor in (3.6) does not exceed a constant times

$$\iint 1_{[0, t_0]}(t) |u(x)|^2 e^{-2\Phi_0(x)} e^{-t^{2k_0+1}|x|^2/C} L(dx) dt, \quad (3.10)$$

for some  $C > 0$ . Here we can carry out the  $t$ -integration first, and to this end, we observe that

$$\int_0^{t_0} e^{-t^{2k_0+1}|x|^2/C} dt = \mathcal{O}(1), \quad \text{for } |x| \leq 1,$$

while for  $|x| \geq 1$ , we get by a change of variables,

$$\int_0^{t_0} e^{-t^{2k_0+1}|x|^2/C} dt \leq \frac{1}{(2k_0+1)|x|^{2/(2k_0+1)}} \int_0^\infty e^{-y/C} y^{-\frac{2k_0}{2k_0+1}} dy.$$

Therefore,

$$\int_0^{t_0} e^{-t^{2k_0+1}|x|^2/C} dt \leq \frac{\mathcal{O}(1)}{(1+|x|^2)^{1/(2k_0+1)}}, \quad x \in \mathbf{C}^n, \quad (3.11)$$

and combining (3.10) and (3.11), we obtain the following bound for the first factor in (3.6),

$$\begin{aligned} & \iint 1_{[0, t_0]}(t) |e^{-tQ/2}u(x)|^2 e^{-2\Phi_0(x)} L(dx) dt \\ & \leq \mathcal{O}(1) \int (1+|x|^2)^{-1/(2k_0+1)} |u(x)|^2 e^{-2\Phi_0(x)} L(dx). \end{aligned} \quad (3.12)$$

To estimate the second factor in (3.6), we observe that all of the analysis developed in Section 2 can be applied also to the operator  $e^{tQ^*}$ , and in particular, we have the bounded operator

$$e^{tQ^*} : H_{\Phi_0}(\mathbf{C}^n) \rightarrow H_{\widehat{\Phi}_t}(\mathbf{C}^n), \quad 0 \leq t \leq t_0, \quad (3.13)$$

where, similarly to (2.25), the quadratic form  $\widehat{\Phi}_t$  satisfies

$$\widehat{\Phi}_t(x) \geq \Phi_0(x) + \frac{t^{2k_0+1}}{C} |x|^2, \quad x \in \mathbf{C}^n, \quad t \in [0, t_0]. \quad (3.14)$$

Combining (3.5), (3.6), (3.12), as well as the analog of the latter estimate involving  $e^{-tQ^*/2}$ , we obtain the estimate

$$|(Ru, v)_{H_{\Phi_0}}| \leq C \|\langle x \rangle^{-\delta} u\|_{L_{\Phi_0}^2} \|\langle x \rangle^{-\delta} v\|_{L_{\Phi_0}^2}, \quad (3.15)$$

where

$$\delta = \frac{1}{2k_0 + 1}. \quad (3.16)$$

Coming back to (3.2), we shall next consider the contribution of the term  $e^{-t_0Q}Q^{-1}$ , and to that end we write as before,

$$(e^{-t_0Q}Q^{-1}u, v)_{H_{\Phi_0}} = (e^{-t_0Q/2}Q^{-1}u, e^{-t_0Q^*/2}v)_{H_{\Phi_0}} = (Q^{-1}e^{-t_0Q/2}u, e^{-t_0Q^*/2}v)_{H_{\Phi_0}}, \quad (3.17)$$

using also the fact that the bounded operators  $e^{-tQ}$  and  $Q^{-1}$  commute. Applying the Cauchy-Schwarz inequality and the fact that  $Q^{-1}$  is bounded on  $H_{\Phi_0}(\mathbf{C}^n)$ , we get

$$|(e^{-t_0Q}Q^{-1}u, v)_{H_{\Phi_0}}| \leq \mathcal{O}(1) \|e^{-t_0Q/2}u\|_{H_{\Phi_0}} \|e^{-t_0Q^*/2}v\|_{H_{\Phi_0}}. \quad (3.18)$$

An application of (3.8) together with Theorem 2.1, along with their natural analogs for the adjoint semigroup  $e^{-tQ^*}$ , see (3.13), (3.14), allows us then to conclude that

$$|(e^{-t_0Q}Q^{-1}u, v)_{H_{\Phi_0}}| \leq \mathcal{O}(1) \|ue^{-\eta|x|^2}\|_{L_{\Phi_0}^2} \|ve^{-\eta|x|^2}\|_{L_{\Phi_0}^2}, \quad (3.19)$$

for some  $\eta > 0$ . Combining (3.2), (3.15), and (3.19), we obtain the basic estimate

$$|(Q^{-1}u, v)_{H_{\Phi_0}}| \leq C \|\langle x \rangle^{-\delta} u\|_{L_{\Phi_0}^2} \|\langle x \rangle^{-\delta} v\|_{L_{\Phi_0}^2}, \quad (3.20)$$

where the factor  $0 < \delta < 1$  has been defined in (3.16). Let us also recall that here  $u, v$  are holomorphic and such that  $\langle x \rangle^N u, \langle x \rangle^N v \in L_{\Phi_0}^2(\mathbf{C}^n)$  for all  $N \in \mathbf{N}$ .

Next, we shall pass to Toeplitz operators, and to that end we would like to replace  $\langle x \rangle^{-\delta} u$  in (3.20) by  $\Pi(\langle x \rangle^{-\delta} u)$ , and similarly for  $\langle x \rangle^{-\delta} v$ , where

$$\Pi : L_{\Phi_0}^2(\mathbf{C}^n) \rightarrow H_{\Phi_0}(\mathbf{C}^n)$$

is the orthogonal projection. Here, given  $q \in L^\infty(\mathbf{C}^n)$ , we introduce the Toeplitz operator with the symbol  $q$ , acting on  $H_{\Phi_0}(\mathbf{C}^n)$ , see [22, Section 1], [25, Chapter 13],

$$\text{Top}(q)u = \Pi(qu), \quad u \in H_{\Phi_0}(\mathbf{C}^n). \quad (3.21)$$

It will be convenient to rewrite (3.20) as follows,

$$|(Q^{-1}u, v)_{H_{\Phi_0}}| \leq C(\lambda) \|(\lambda + |x|^2)^{-\delta/2}u\|_{L_{\Phi_0}^2} \|(\lambda + |x|^2)^{-\delta/2}v\|_{L_{\Phi_0}^2}, \quad (3.22)$$

where  $\lambda \geq 1$  is to be chosen sufficiently large but fixed, and the constant  $C(\lambda) > 0$  depends on  $\lambda$ . Using that  $\Pi u = u$ , we get

$$\|(\lambda + |x|^2)^{-\delta/2}u\|_{L_{\Phi_0}^2} \leq \|\Pi(\lambda + |x|^2)^{-\delta/2}u\|_{H_{\Phi_0}} + \|\Pi, (\lambda + |x|^2)^{-\delta/2}\|u\|_{L_{\Phi_0}^2}. \quad (3.23)$$

Here we can write  $\Pi(\lambda + |x|^2)^{-\delta/2}u = \text{Top}((\lambda + |x|^2)^{-\delta/2})u$ , see (3.21). We next claim that

$$K := [\Pi, (\lambda + |x|^2)^{-\delta/2}](\lambda + |x|^2)^{(\delta+1)/2} = \mathcal{O}(1) : L_{\Phi_0}^2(\mathbf{C}^n) \rightarrow L_{\Phi_0}^2(\mathbf{C}^n), \quad (3.24)$$

uniformly in  $\lambda \geq 1$ . When verifying the claim, let us recall from [22, Section 1], [11, Proposition 1.3.4], that the operator  $\Pi$  is given by

$$\Pi u(x) = C \int e^{2\psi_0(x, \bar{y})} u(y) e^{-2\Phi_0(y)} L(dy), \quad C > 0,$$

where  $\psi_0(x, y)$  is the unique holomorphic quadratic form on  $\mathbf{C}_x^n \times \mathbf{C}_y^n$ , such that  $\psi_0(x, \bar{x}) = \Phi_0(x)$ . We have the basic property,

$$2\text{Re } \psi_0(x, \bar{y}) - \Phi_0(x) - \Phi_0(y) \sim -|x - y|^2, \quad (3.25)$$

on  $\mathbf{C}_x^n \times \mathbf{C}_y^n$ , reflecting the strict plurisubharmonicity of  $\Phi_0$ , see [11, Section 1.3]. Letting  $\Pi(x, y)$  stand for the integral kernel of  $\Pi$ , we have therefore,

$$|\Pi(x, y)| e^{\Phi_0(y) - \Phi_0(x)} \leq \mathcal{O}(1) e^{-|x-y|^2/C}, \quad C > 0. \quad (3.26)$$

The integral kernel of the operator  $K$  in (3.24) is given by

$$K(x, y) = \Pi(x, y) \left( (\lambda + |y|^2)^{-\delta/2} - (\lambda + |x|^2)^{-\delta/2} \right) (\lambda + |y|^2)^{(\delta+1)/2}, \quad (3.27)$$

and when estimating it, we observe that

$$|(\lambda + |y|^2)^{-\delta/2} - (\lambda + |x|^2)^{-\delta/2}| \leq \mathcal{O}(1) |x - y| \int_0^1 \frac{dt}{(\lambda + |tx + (1-t)y|^2)^{(\delta+1)/2}}, \quad (3.28)$$



where  $\mathcal{O}(1)$  is uniform in  $\lambda \geq 1$ . The standard inequality

$$\frac{1 + |x|}{1 + |y|} \leq 1 + |x - y|$$

shows next that uniformly in  $t \in [0, 1]$  and  $\lambda \geq 1$ , we have

$$\left( \frac{\lambda + |y|^2}{\lambda + |tx + (1-t)y|^2} \right)^{1/2} \leq 2(1 + |x - y|). \quad (3.29)$$

Combining (3.26), (3.28), and (3.29), we conclude that the absolute value of the reduced kernel  $e^{-\Phi_0(x)} K(x, y) e^{\Phi_0(y)}$  of the operator  $K$  in (3.24) does not exceed

$$\mathcal{O}(1) e^{-|x-y|^2/C} |x - y| (1 + |x - y|)^{\delta+1},$$

uniformly in  $\lambda \geq 1$ . An application of Schur's lemma shows that (3.24) holds.

Combining (3.23) and (3.24), we get

$$\|(\lambda + |x|^2)^{-\delta/2} u\|_{L^2_{\Phi_0}} \leq \|\text{Top}((\lambda + |x|^2)^{-\delta/2}) u\|_{H_{\Phi_0}} + \mathcal{O}(\lambda^{-1/2}) \|(\lambda + |x|^2)^{-\delta/2} u\|_{L^2_{\Phi_0}},$$

and choosing  $\lambda > 1$  sufficiently large but fixed, we conclude that

$$\|(\lambda + |x|^2)^{-\delta/2} u\|_{L^2_{\Phi_0}} \leq \mathcal{O}(1) \|\text{Top}((\lambda + |x|^2)^{-\delta/2}) u\|_{H_{\Phi_0}}. \quad (3.30)$$

The parameter  $\lambda$  will be kept fixed from now on and the dependence on  $\lambda$  will not be indicated explicitly. Injecting the estimate (3.30) back into (3.22), we obtain that

$$|(Q^{-1}u, v)_{H_{\Phi_0}}| \leq C \|\text{Top}((\lambda + |x|^2)^{-\delta/2}) u\|_{H_{\Phi_0}} \|\text{Top}((\lambda + |x|^2)^{-\delta/2}) v\|_{H_{\Phi_0}}. \quad (3.31)$$

We shall now return to the real side by undoing the FBI transform. When doing so, let us recall from [22, Section 1] as well as from [25, Theorems 13.9, 13.10], that there exists  $a \in C^\infty(\mathbf{R}^{2n})$  such that for all  $\alpha \in \mathbf{N}^{2n}$ , we have

$$\partial^\alpha a(X) = \mathcal{O}_\alpha(1) m(X), \quad m(X) = \langle X \rangle^{-\delta}, \quad (3.32)$$

and such that

$$\text{Top}((\lambda + |x|^2)^{-\delta/2}) = T a^w(y, D_y) T^{-1}. \quad (3.33)$$

Indeed, it follows from [22, Section 1] that we have (3.33) with  $a = b \circ \kappa_T$ , where  $b \in S(\Lambda_{\Phi_0})$  is such that

$$b\left(x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x}(x)\right) = \left(\exp\left(\frac{1}{4} (\partial_x \partial_{\bar{x}} \Phi_0)^{-1} \partial_x \cdot \partial_{\bar{x}}\right) q\right)(x), \quad q(x) = (\lambda + |x|^2)^{-\delta/2}.$$

It follows that when  $u, v \in \mathcal{S}(\mathbf{R}^n)$ , we get from (3.31),

$$|((q^w)^{-1}u, v)_{L^2}| \leq C \|a^w(x, D_x)u\|_{L^2} \|a^w(x, D_x)v\|_{L^2}. \quad (3.34)$$

It will be convenient to rewrite (3.34) using the positive selfadjoint operator  $\Lambda = (1 + x^2 + D_x^2)^{1/2}$  on  $L^2(\mathbf{R}^n)$ , defined by means of the functional calculus for the harmonic oscillator. To that end, let us set

$$\Gamma = \frac{dx^2 + d\xi^2}{\langle X \rangle^2}, \quad X = (x, \xi) \in \mathbf{R}^{2n},$$

and introduce the corresponding symbol class

$$S(\langle X \rangle^m, \Gamma) = \{a \in C^\infty(\mathbf{R}^{2n}); |\partial^\alpha a(X)| \leq C_\alpha \langle X \rangle^{m-|\alpha|}, \alpha \in \mathbf{N}^{2n}\}, \quad m \in \mathbf{R}.$$

Let us recall from [5, Theorem 1.11.1] that

$$\Lambda^r \in \text{Op}^w(S(\langle X \rangle^r, \Gamma)), \quad r \in \mathbf{R}. \quad (3.35)$$

Using (3.32), (3.35), and the Calderón-Vaillancourt theorem, we conclude that the operator

$$a^w(x, D_x)\Lambda^\delta : L^2(\mathbf{R}^n) \rightarrow L^2(\mathbf{R}^n)$$

is bounded. We get therefore from (3.34), for  $u, v \in \mathcal{S}(\mathbf{R}^n)$ ,

$$|((q^w)^{-1}u, v)_{L^2}| \leq C \|\Lambda^{-\delta}u\|_{L^2} \|\Lambda^{-\delta}v\|_{L^2}. \quad (3.36)$$

Using that  $\Lambda^\delta$  is a bijection on  $\mathcal{S}(\mathbf{R}^n)$  and rewriting (3.36) in the form

$$|((q^w)^{-1}u, \Lambda^\delta v)_{L^2}| \leq C \|\Lambda^{-\delta}u\|_{L^2} \|v\|_{L^2}, \quad u, v \in \mathcal{S}(\mathbf{R}^n), \quad (3.37)$$

we observe that (3.37) extends to all  $u \in L^2(\mathbf{R}^n)$  and  $v \in \mathcal{D}(\Lambda^\delta) \subseteq L^2(\mathbf{R}^n)$ , since  $\mathcal{S}(\mathbf{R}^n)$  is dense in  $\mathcal{D}(\Lambda^\delta)$  with respect to the graph norm. The estimate

$$|((q^w)^{-1}u, \Lambda^\delta v)_{L^2}| \leq C \|\Lambda^{-\delta}u\|_{L^2} \|v\|_{L^2}, \quad u \in L^2(\mathbf{R}^n), v \in \mathcal{D}(\Lambda^\delta)$$

implies that  $\mathcal{D}(q^w) \subset \mathcal{D}(\Lambda^\delta)$ , with

$$\|\Lambda^\delta(q^w)^{-1}u\|_{L^2} \leq C \|\Lambda^{-\delta}u\|_{L^2}, \quad u \in L^2(\mathbf{R}^n). \quad (3.38)$$

Using the bound (3.38), it is now easy to conclude the proof of Theorem 1.1. It follows from (3.38) that

$$\|\Lambda^\delta v\|_{L^2} \leq C \|\Lambda^{-\delta} q^w v\|_{L^2}, \quad v \in \mathcal{S}(\mathbf{R}^n), \quad (3.39)$$

and taking  $v = \Lambda^\delta u$ ,  $u \in \mathcal{S}(\mathbf{R}^n)$ , we obtain in view of (3.39),

$$\|\Lambda^{2\delta} u\|_{L^2} \leq C (\|q^w u\|_{L^2} + \|\Lambda^{-\delta} [q^w, \Lambda^\delta] u\|_{L^2}) \leq C (\|q^w u\|_{L^2} + \|u\|_{L^2}). \quad (3.40)$$

Here we have used that since  $q$  is quadratic, the Weyl symbol of the operator  $[q^w, \Lambda^\delta]$  is of the form  $i^{-1} H_q b \in S(\langle X \rangle^\delta, \Gamma)$ , for some  $b \in S(\langle X \rangle^\delta, \Gamma)$ , and therefore the operator  $\Lambda^{-\delta} [q^w, \Lambda^\delta] \in \text{Op}^w(S(1, \Gamma))$  is bounded on  $L^2(\mathbf{R}^n)$ . The proof of Theorem 1.1 is complete.

*Remark.* The arguments developed in this section allow us to establish the following subelliptic resolvent estimate, generalizing (3.40): there exists a constant  $C > 0$  such that for all  $u \in \mathcal{D}(q^w)$  and all  $\lambda \in \mathbf{R}$ , we have

$$\|\Lambda^{2\delta} u\|_{L^2} \leq C (\|(q^w - i\lambda)u\|_{L^2} + \|u\|_{L^2}). \quad (3.41)$$

Indeed, in order to obtain (3.41), we inspect the arguments of the present section, concluding that everything works as above, leading to (3.41), provided that we take as our starting point the representation of the resolvent along the imaginary axis as the Fourier transform of the semigroup,

$$(Q - i\lambda)^{-1} = \int_0^\infty e^{-t(Q-i\lambda)} dt, \quad \lambda \in \mathbf{R},$$

and observe that thanks to the exponential decay of the norm of the semigroup  $e^{-tQ}$ , as  $t \rightarrow \infty$ , established in [8, Theorem 1.2.3], we have

$$\|(Q - i\lambda)^{-1}\|_{\mathcal{L}(H_{\Phi_0}(\mathbf{C}^n), H_{\Phi_0}(\mathbf{C}^n))} \leq \mathcal{O}(1),$$

uniformly in  $\lambda \in \mathbf{R}$ . As explained in [15, Section 3] and [16, Subsection 3.2], the subelliptic estimate (3.41) leads directly to some accurate resolvent estimates for  $q^w$ , in parabolic regions near the imaginary axis.

## 4 Smoothing estimates for the semigroup

In this section, we shall establish Theorem 1.2. When doing so, we shall first continue to work on the FBI transform side, and let us recall from (2.53), (2.54), that we have a bounded operator

$$e^{-tQ} : H_{\Phi_0}(\mathbf{C}^n) \rightarrow H_{\tilde{\Phi}_t}(\mathbf{C}^n), \quad (4.1)$$

for all  $t \in [0, t_0]$ , with  $t_0 > 0$  sufficiently small, with the operator norm in (4.1) being bounded uniformly in  $t \in [0, t_0]$ . Here  $\tilde{\Phi}_t$  is a real strictly plurisubharmonic quadratic form on  $\mathbf{C}^n$  such that

$$\tilde{\Phi}_t(x) \leq \Phi_0(x) - \frac{t^{2k_0+1}}{C} |x|^2, \quad x \in \mathbf{C}^n, \quad t \in [0, t_0]. \quad (4.2)$$

Let  $p_0(x, \xi) = x^2 + \xi^2$  be the symbol of the harmonic oscillator, and as in (2.9), set  $q_0 = p_0 \circ \kappa_T^{-1}$ . In what follows, we shall rely only on the observation that  $q_0$  is a holomorphic quadratic form on  $\mathbf{C}^{2n}$  such that its restriction to  $\Lambda_{\Phi_0}$  is real positive definite. Let us set  $Q_0 = q_0^w(x, D_x)$ . The quadratic differential operator  $Q_0$  is selfadjoint on  $H_{\Phi_0}(\mathbf{C}^n)$ , with the domain

$$\mathcal{D}(Q_0) = \{u \in H_{\Phi_0}(\mathbf{C}^n); (1 + |x|^2)u \in L^2_{\Phi}(\mathbf{C}^n)\}.$$

Proceeding similarly to the discussion in Section 2, we shall now consider the operator  $e^{sQ_0}$ , for  $s \in [0, s_0]$ , with  $s_0 > 0$  small enough, acting on  $H_{\tilde{\Phi}_t}(\mathbf{C}^n)$ . Studying the evolution problem

$$(\partial_s - Q_0)u(s, x) = 0, \quad u|_{s=0} = u_0 \in H_{\tilde{\Phi}_t}(\mathbf{C}^n),$$

and arguing as in Section 2, we see that for  $s \in [0, s_0]$ , with  $s_0 > 0$  small enough, and all  $t \in [0, t_0]$ , the operator  $e^{sQ_0}$  is bounded

$$e^{sQ_0} : H_{\tilde{\Phi}_t}(\mathbf{C}^n) \rightarrow H_{\tilde{\Phi}_{t,s}}(\mathbf{C}^n). \quad (4.3)$$

Here  $\tilde{\Phi}_{t,s}$  is a strictly plurisubharmonic quadratic form on  $\mathbf{C}^n$ , depending smoothly on  $t \geq 0$  and  $s \geq 0$  small enough, such that

$$\frac{\partial \tilde{\Phi}_{t,s}}{\partial s}(x) - \operatorname{Re} q_0 \left( x, \frac{2}{i} \frac{\partial \tilde{\Phi}_{t,s}}{\partial x}(x) \right) = 0, \quad \tilde{\Phi}|_{t,s=0} = \tilde{\Phi}_t. \quad (4.4)$$

It follows that

$$\tilde{\Phi}_{t,s}(x) = \tilde{\Phi}_t(x) + \mathcal{O}(s|x|^2), \quad (4.5)$$

uniformly for  $t \in [0, t_0]$ , with  $t_0 > 0$  sufficiently small. Choosing

$$s = s(t) = \frac{t^{2k_0+1}}{C_0}, \quad (4.6)$$

where the constant  $C_0$  is large enough, we conclude, in view of (4.2), (4.3), and (4.5), that for all  $t \in [0, t_0]$ , the operator  $e^{s(t)Q_0}$  is bounded,

$$e^{s(t)Q_0} : H_{\tilde{\Phi}_t}(\mathbf{C}^n) \rightarrow H_{\Phi_0}(\mathbf{C}^n). \quad (4.7)$$

Combining this observation with (4.1), we conclude that for all  $t \in [0, t_0]$ , the composition

$$e^{s(t)Q_0} e^{-tQ} : H_{\Phi_0}(\mathbf{C}^n) \rightarrow H_{\Phi_0}(\mathbf{C}^n), \quad (4.8)$$

is bounded, with the operator norm being bounded uniformly, for  $t \in [0, t_0]$ . For future reference, coming back to the real side, let us summarize the discussion so far in the following result.

**Proposition 4.1** *Let  $q : \mathbf{R}_x^n \times \mathbf{R}_\xi^n \rightarrow \mathbf{C}$  be a quadratic form with  $\operatorname{Re} q \geq 0$ , such that (1.7) holds, and let us define  $k_0 \in \mathbf{N}$  as in (1.8). There exist  $C_0 > 0$  and  $t_0 > 0$  such that for all  $t \in [0, t_0]$ , we have a bounded operator*

$$e^{\frac{t^{2k_0+1}}{C_0}(D_x^2+x^2)} e^{-tq^w} : L^2(\mathbf{R}^n) \rightarrow L^2(\mathbf{R}^n),$$

with the norm in  $\mathcal{L}(L^2(\mathbf{R}^n), L^2(\mathbf{R}^n))$  bounded uniformly, for  $t \in [0, t_0]$ .

*Remark.* The discussion in Section 2 shows that the analogs of (4.1), (4.2) are also valid for the adjoint semigroup,

$$e^{-tQ^*} : H_{\Phi_0}(\mathbf{C}^n) \rightarrow H_{\Phi_0}(\mathbf{C}^n), \quad t \in [0, t_0],$$

for  $t_0 > 0$  small enough, and therefore, an application of Proposition 4.1 shows that we have a bounded operator

$$e^{\frac{t^{2k_0+1}}{C_0}(D_x^2+x^2)} e^{-t\bar{q}^w} : L^2(\mathbf{R}^n) \rightarrow L^2(\mathbf{R}^n),$$

uniformly for  $t \in [0, t_0]$ . Let us introduce the domain  $\mathcal{D} = \mathcal{D}(e^{\frac{t^{2k_0+1}}{C_0}(D_x^2+x^2)}) \subseteq L^2(\mathbf{R}^n)$  of the unbounded selfadjoint operator  $e^{\frac{t^{2k_0+1}}{C_0}(D_x^2+x^2)}$  on  $L^2(\mathbf{R}^n)$ , and notice that  $v \in L^2(\mathbf{R}^n)$  belongs to  $\mathcal{D}$  precisely when

$$(v, \psi_\alpha)_{L^2} \exp\left(\frac{2t^{2k_0+1}}{C_0} |\alpha|\right) \in l^2(\mathbf{N}^n).$$

Here  $\psi_\alpha$  are the Hermite functions, see (4.12) below. In particular, we notice that  $\mathcal{D}$  is dense in  $L^2(\mathbf{R}^n)$ , since it contains all finite sums of the Hermite functions. When  $u \in L^2(\mathbf{R}^n)$  and  $v \in \mathcal{D}$ , we write for  $t \in [0, t_0]$ ,

$$\left| \left( e^{\frac{t^{2k_0+1}}{C_0}(D_x^2+x^2)} e^{-t\bar{q}^w} u, v \right)_{L^2} \right| \leq \mathcal{O}(1) \|u\|_{L^2} \|v\|_{L^2}. \quad (4.9)$$

The scalar product in the left hand side in (4.9) is equal to

$$(u, e^{-tq^w} e^{\frac{t^{2k_0+1}}{C_0}(D_x^2+x^2)} v)_{L^2},$$

and therefore we get

$$\| e^{-tq^w} e^{\frac{t^{2k_0+1}}{C_0}(D_x^2+x^2)} v \|_{L^2} \leq \mathcal{O}(1) \| v \|_{L^2}, \quad t \in [0, t_0].$$

Using the density of  $\mathcal{D}$  in  $L^2(\mathbf{R}^n)$ , we conclude that the following operator is bounded on  $L^2(\mathbf{R}^n)$ , uniformly for  $t \in [0, t_0]$ ,

$$e^{-tq^w} e^{\frac{t^{2k_0+1}}{C_0}(D_x^2+x^2)} : L^2(\mathbf{R}^n) \rightarrow L^2(\mathbf{R}^n).$$

Returning to (4.8), we shall now estimate the norm of the operator  $Q_0^N e^{-tQ}$ , viewed as a bounded operator on  $H_{\Phi_0}(\mathbf{C}^n)$ , when  $N \in \mathbf{N}$ . Writing

$$Q_0^N e^{-tQ} = Q_0^N e^{-s(t)Q_0} e^{s(t)Q_0} e^{-tQ},$$

where the operators involved are quadratic Fourier integral operators in the complex domain, and recalling the uniform boundedness of (4.8), we obtain that there exists a constant  $C > 0$  such that for all  $t \in [0, t_0]$  we have

$$\| Q_0^N e^{-tQ} \|_{\mathcal{L}(H_{\Phi_0}(\mathbf{C}^n), H_{\Phi_0}(\mathbf{C}^n))} \leq C \| Q_0^N e^{-s(t)Q_0} \|_{\mathcal{L}(H_{\Phi_0}(\mathbf{C}^n), H_{\Phi_0}(\mathbf{C}^n))}. \quad (4.10)$$

By the selfadjoint functional calculus, we have

$$\| Q_0^N e^{-s(t)Q_0} \|_{\mathcal{L}(H_{\Phi_0}(\mathbf{C}^n), H_{\Phi_0}(\mathbf{C}^n))} \leq \sup_{\lambda \geq 0} (\lambda^N e^{-s(t)\lambda}) \leq \frac{N!}{s(t)^N}, \quad (4.11)$$

and we get the smoothing estimate

$$\| Q_0^N e^{-tQ} \|_{\mathcal{L}(H_{\Phi_0}(\mathbf{C}^n), H_{\Phi_0}(\mathbf{C}^n))} \leq \frac{C^{N+1} N!}{t^{(2k_0+1)N}}, \quad 0 < t \leq t_0,$$

valid for some  $C > 0$  and all  $N \in \mathbf{N}$ . Undoing the FBI transform and coming back to the  $L^2(\mathbf{R}^n)$ -side, we see that we have proved the bound (1.14) in Theorem 1.2.

*Remark.* The work [1] also examines the semigroup  $e^{-tq^w}$  on the FBI transform side, but focusing on a particular FBI transform and an FBI-side harmonic oscillator  $P_0$  adapted to the symbol  $q$ . Combining Theorems 2.10, 3.8, and 4.8 in [1] gives a result similar to Proposition 4.1, which reads that

$$e^{\frac{t^{2k_0+1}}{C_0} P_0} e^{-tQ} \in \mathcal{L}(H_{\Phi_0}(\mathbf{C}^n), H_{\Phi_0}(\mathbf{C}^n)), \quad t \in [0, t_0], \quad 0 < t_0 \ll 1,$$

and that  $P_0$  may be replaced by the usual harmonic oscillator on the real side. Examining the proof of Theorem 2.10 shows that the operator norm of this composition is  $\mathcal{O}(1)$ , for  $t \geq 0$  sufficiently small.

It remains for us to finish the proof of Theorem 1.2, by deriving the Gelfand-Shilov bounds (1.15) on the heat semigroup  $e^{-tq^w}$ . When doing so, we shall combine Proposition 4.1 with some arguments of [14]. Let us write  $P = D_x^2 + x^2$  on  $L^2(\mathbf{R}^n)$ , and let us recall that the spectrum of  $P$  is given by the eigenvalues  $\lambda_\alpha = 2|\alpha| + n$ ,  $\alpha \in \mathbf{N}^n$ , and the corresponding eigenfunctions are the Hermite functions,

$$\psi_\alpha(x) = H_\alpha(x)e^{-x^2/2}, \quad \alpha \in \mathbf{N}^n, \quad (4.12)$$

forming an orthonormal basis of  $L^2(\mathbf{R}^n)$ . Here the Hermite polynomials  $H_\alpha(x)$  satisfy

$$H_\alpha(x) = \prod_{j=1}^n H_{\alpha_j}(x_j).$$

Let  $u \in L^2(\mathbf{R}^n)$ . Setting  $u(t) = e^{-tq^w}u$ ,  $t \geq 0$ , we can write

$$u(t) = \sum_{\alpha} a_{\alpha}(t) \psi_{\alpha}, \quad a_{\alpha}(t) = (u(t), \psi_{\alpha})_{L^2}. \quad (4.13)$$

An application of Proposition 4.1 shows that

$$\sum_{\alpha} |a_{\alpha}(t)|^2 e^{\frac{2t^{2k_0+1}}{C_0} \lambda_{\alpha}} \leq \mathcal{O}(1) \|u\|_{L^2(\mathbf{R}^n)}^2,$$

for all  $t \in [0, t_0]$ , so that for all  $\alpha \in \mathbf{N}^n$ , we have

$$|a_{\alpha}(t)| \leq \mathcal{O}(1) e^{-\frac{2t^{2k_0+1}|\alpha|}{C_0}} \|u\|_{L^2(\mathbf{R}^n)}. \quad (4.14)$$

Using [14, Lemma A.1], we observe next that there exists a constant  $C > 0$  such that for all  $\varepsilon \in (0, 1/2)$  and all  $\mu, \nu \in \mathbf{N}^n$ , we have

$$\|x^{\mu} \partial_x^{\nu} \psi_{\alpha}\|_{L^2(\mathbf{R}^n)} \leq C^{1+|\mu|+|\nu|} (\mu!)^{1/2} (\nu!)^{1/2} \frac{e^{\varepsilon|\alpha|}}{\varepsilon^{(|\mu|+|\nu|)/2}}. \quad (4.15)$$

Combining (4.13), (4.14), and (4.15), we get

$$\|x^{\mu} \partial_x^{\nu} u(t)\|_{L^2(\mathbf{R}^n)} \leq \frac{C^{1+|\mu|+|\nu|} (\mu!)^{1/2} (\nu!)^{1/2}}{\varepsilon^{(|\mu|+|\nu|)/2}} \sum_{\alpha} \exp\left(\varepsilon|\alpha| - \frac{2t^{2k_0+1}}{C_0} |\alpha|\right) \|u\|_{L^2(\mathbf{R}^n)}.$$

Choosing

$$\varepsilon = \frac{t^{2k_0+1}}{C_0},$$

we conclude that with a new constant  $C$ ,

$$\|x^\mu \partial_x^\nu u(t)\|_{L^2(\mathbf{R}^n)} \leq \frac{C^{1+|\mu|+|\nu|} (\mu!)^{1/2} (\nu!)^{1/2}}{t^{\frac{(2k_0+1)}{2}(|\mu|+|\nu|)}} F\left(\frac{t^{2k_0+1}}{C_0}\right) \|u\|_{L^2(\mathbf{R}^n)}, \quad 0 < t \leq t_0, \quad (4.16)$$

for  $t_0 > 0$  sufficiently small. Here

$$F(y) = \sum_{\alpha} e^{-y|\alpha|}, \quad y > 0.$$

When estimating  $F(y)$  as  $y \rightarrow 0$ , we notice that

$$F(y) = \frac{1}{(n-1)!} \sum_{m=0}^{\infty} (m+1) \dots (m+n-1) e^{-ym} \leq C_n \left(1 + \sum_{m=0}^{\infty} m^{n-1} e^{-ym}\right), \quad (4.17)$$

where the constant  $C_n > 0$  depends on  $n$  only. Here

$$\sum_{m=0}^{\infty} m^{n-1} e^{-ym} = \left(-\frac{d}{dy}\right)^{n-1} \sum_{m=0}^{\infty} e^{-ym} = \left(-\frac{d}{dy}\right)^{n-1} \frac{e^y}{e^y - 1},$$

and an elementary argument allows us therefore to conclude that

$$F(y) \leq \mathcal{O}_n(1) y^{-n}, \quad 0 < y \leq 1. \quad (4.18)$$

Combining (4.16) and (4.18), we get

$$\|x^\mu \partial_x^\nu u(t)\|_{L^2(\mathbf{R}^n)} \leq \frac{C^{1+|\mu|+|\nu|} (\mu!)^{1/2} (\nu!)^{1/2}}{t^{\frac{(2k_0+1)}{2}(|\mu|+|\nu|+2n)}} \|u\|_{L^2(\mathbf{R}^n)}, \quad 0 < t \leq t_0. \quad (4.19)$$

To pass to the  $L^\infty$ -norms, it suffices to apply the Sobolev embedding theorem,

$$\|x^\mu \partial_x^\nu u(t)\|_{L^\infty(\mathbf{R}^n)} \leq \mathcal{O}(1) \sum_{|\alpha| \leq s} \|D^\alpha (x^\mu \partial_x^\nu u(t))\|_{L^2(\mathbf{R}^n)}, \quad (4.20)$$

where  $s > n/2$  is an integer. The estimate (1.15) follows from (4.19) and (4.20), and this completes the proof of Theorem 1.2.



# References

- [1] A. Aleman and J. Viola, *On weak and strong solution operators for evolution equations coming from quadratic operators*, J. Spectral Theory, to appear.
- [2] E. Caliceti, S. Graffi, M. Hitrik, and J. Sjöstrand, *Quadratic  $\mathcal{PT}$ -symmetric operators with real spectrum and similarity to self-adjoint operators*, J. Phys. A: Math. Theor., **45** (2012), 444007.
- [3] M. Dimassi and J. Sjöstrand, *Spectral asymptotics in the semi-classical limit*, Cambridge University Press, 1999.
- [4] S. Gadat and L. Miclo, *Spectral decompositions and  $L^2$ -operator norms of toy hypocoercive semi-groups*, Kinet. Relat. Models, **6** (2013), 317–372.
- [5] B. Helffer, *Théorie spectrale pour des opérateurs globalement elliptiques*, Astérisque, 112. Société Mathématique de France, Paris, 1984.
- [6] B. Helffer and F. Nier, *Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten laplacians*, SLN 1862, Springer Verlag, 2005.
- [7] F. Hérau, J. Sjöstrand, and C. Stolk, *Semiclassical analysis for the Kramers-Fokker-Planck equation*, Comm. PDE, **30** (2005), 689–760.
- [8] M. Hitrik and K. Pravda-Starov, *Spectra and semigroup smoothing for non-elliptic quadratic operators*, Math. Ann. **344** (2009), 801–846.
- [9] M. Hitrik and K. Pravda-Starov, *Eigenvalues and subelliptic estimates for non-selfadjoint semiclassical operators with double characteristics*, Annales Inst. Fourier, **63** (2013), 985–1032.
- [10] M. Hitrik, K. Pravda-Starov, and J. Viola, *Short-time asymptotics of the regularizing effect for semigroups generated by quadratic operators*, preprint, 2015.
- [11] M. Hitrik and J. Sjöstrand, *Two minicourses on analytic microlocal analysis*, ”Algebraic and Analytic Microlocal Analysis”, M. Hitrik, D. Tamarkin, B. Tsygan, and S. Zelditch, eds. Springer, to appear.
- [12] M. Hitrik, J. Sjöstrand, and J. Viola, *Resolvent estimates for elliptic quadratic differential operators*, Anal. PDE **6** (2013), 181196.
- [13] L. Hörmander, *Symplectic classification of quadratic forms, and general Mehler formulas*, Math. Z., **219** (1995), 413–449.

- [14] N. Lerner, Y. Morimoto, K. Pravda-Starov, and C.-J. Xu, *Gelfand-Shilov and Gevrey smoothing effect for the spatially inhomogeneous non-cutoff Kac equation*, J. Funct. Anal. **269** (2015), 459–535.
- [15] F. Nier, *Boundary conditions and subelliptic estimates for geometric Kramers-Fokker-Planck operators on manifolds with boundaries*, preprint, 2014, <http://arxiv.org/abs/1309.5070>.
- [16] M. Ottobre, G. Pavliotis, and K. Pravda-Starov, *Exponential return to equilibrium for hypoelliptic quadratic systems*, J. Funct. Anal. **262** (2012), 4000–4039.
- [17] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.
- [18] K. Pravda-Starov, *Subelliptic estimates for quadratic differential operators*, American J. Math. **133** (2011), 39–89.
- [19] J. Sjöstrand, *Singularités analytiques microlocales*, Astérisque, **95** (1982), 1–166, Soc. Math. France, Paris.
- [20] J. Sjöstrand, *Analytic wavefront sets and operators with multiple characteristics*, Hokkaido Math. Journal, **12**, no. 3, part 2, 392–433 (1983).
- [21] J. Sjöstrand, *Geometric bounds on the density of resonances for semiclassical problems*, Duke Math. J. **60** (1990), 1–57.
- [22] J. Sjöstrand, *Function spaces associated to global I-Lagrangian manifolds*, Structure of solutions of differential equations, Katata/Kyoto, 1995, World Sci. Publ., River Edge, NJ (1996).
- [23] J. Sjöstrand, *Resolvent estimates for non-selfadjoint operators via semigroups*, Around the research of Vladimir Maz'ya. III, 359384, Int. Math. Ser. (N. Y.), 13, Springer, New York, 2010.
- [24] J. Viola, *Spectral projections and resolvent bounds for partially elliptic quadratic differential operators*, J. Pseudo-Differ. Oper. Appl. **4** (2013), 145–221.
- [25] M. Zworski, *Semiclassical analysis*, American Math. Society, 2012.